

A Hamilton-Jacobi equation approach to Large Deviation of Markov Processes

Jin Feng (joint work with Tom Kurtz, U. Wisconsin)

Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

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Essence of my talk



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- Left hand: calculus of variation
- Right hand: probability
- A poor man's version of "everything about Large Deviation"

Principles of Laplace v.s. Large Deviation

- The Laplace principle

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- LDP, a functional level re-formulation,

$$\begin{aligned} \lim_n \frac{1}{n} \log E[e^{nf(X_n)}] &= \lim_n \frac{1}{n} \log \int e^{n(f-I)(x)} \pi(dx) \\ &= \sup_{x \in S} \{f(x) - I(x)\}, \quad \forall f. \end{aligned}$$

Pressure-Entropy duality, the nature of large deviation

- Entropy-Pressure duality à la Gibbs:

$$\log \int_S e^{g(z)} P(dz) = \sup_Q \{ \langle g, Q \rangle - R(Q||P) \}$$

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$$\log \int_S e^{g(z)} P(dz) = \sup_Q \{ \langle g, Q \rangle - R(Q \| P) \}$$

- Renormalize: let $dP^g := \frac{e^g}{Z_g} dP$, then

$$\sup_Q \{ \langle g, Q \rangle - R(Q \| P) \} = - \inf_Q R(Q \| P^g) + \log Z_g.$$

Large Deviation - Rigorous definition

Let $\{X_n : n = 1, 2, \dots\}$ be metric space S -valued r.v.s.

- There exists lower semicontinuous $I : S \mapsto [0, +\infty]$ satisfying

$$\begin{aligned} - \inf_{x \in A^\circ} I(x) &\leq \liminf_n \frac{1}{n} \log P(X_n \in A^\circ) \\ &\leq \limsup_n \frac{1}{n} \log P(X_n \in \bar{A}) \leq - \inf_{x \in \bar{A}} I(x). \end{aligned}$$

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Exponential Tightness

- For each $a > 0$, there exists a compact $K_a \subset\subset S$ such that

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- Equivalently (S complete separable metric space), for each $a > 0$ and $\epsilon > 0$, there exists $K_{a,\epsilon} \subset\subset S$,

$$\limsup_n \frac{1}{n} \log P(X_n \notin K_{a,\epsilon}^\epsilon) \leq -a$$

The Laplace Principle

$\{X_n : n = 1, 2, \dots\}$ is S -valued (S is Polish)

- (Varadhan) $\{X_n : n = 1, 2, \dots\}$ satisfies LDP with good rate function I , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log E[e^{nf(X_n)}] = \sup_{x \in S} \{f(x) - I(x)\}.$$

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- (Bryc) Let $\{X_n : n = 1, 2, \dots\}$ be exponentially tight,

$$\Lambda(f) = \lim_n \frac{1}{n} \log E[e^{nf(X_n)}], \quad f \in D \subset C_b(S).$$

Then $\{X_n : n = 1, 2, \dots\}$ satisfies LDP with good rate function $I(x) = \sup_{f \in C_b(S)} \{f(x) - \Lambda(f)\}$.

Smaller class of test functions $D \subset C_b(S)$

Definition: $D \subset C_b(S)$ is rate function determining if

$$\sup_{f \in D} \{f(x) - \Lambda(f)\} = \sup_{f \in C_b(S)} \{f(x) - \Lambda(f)\}.$$

- Useful test functions are of the form $-md(x, \cdot)$...

In the following talk,

- $X_n(\cdot)$ is a sequence of Markov processes, hence

$$S := D_E[0, \infty).$$

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3. $\eta_n : E_n \mapsto E$ and $X_n(t) := \eta_n(Y_n(t)) = \eta_n \circ Y_n(t)$ is E -valued process;

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■ Main questions:

1. Can we derive large deviation behavior of $\{X_n(\cdot) : n = 1, 2, \dots\}$ based on information in sequence of operators $\{A_n : n = 1, 2, \dots\}$?
2. What is its connection with variational problems?

Large deviation for Markov processes $\{Y_n(\cdot) : n = 1, 2, \dots\}$

Take the case $E = E_n$ and $\eta_n = I$, so

$$X_n(t) := \eta_n(Y_n(t)) = Y_n(t):$$

- One step transition determines everything

$$(V_n(t)f)(x) := \frac{1}{n} \log E[e^{nf(X_n(t))} | X_n(0) = x].$$

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- V_n is the Nisio semigroup by entropy-pressure dual

$$V_n(t)f(x) = \sup_{Q_n^x} E^{Q_n^x}[f(X_n(t)) - \log \frac{dQ_n^x}{dP_n^x} |_{\mathcal{F}_t^n}(X_n(\cdot))]$$

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- What happened infinitesimally?

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- Here is a program for LDP:

$$1. (H_n \rightarrow H) + 2. (\text{exp. tight.}) + 3. (\text{HJ equations}) \\ \Rightarrow \text{LDP.}$$

Large deviation for Markov processes $\{Y_n(\cdot) : n = 1, 2, \dots\}$

- Rate function? $V_n \rightarrow V$ is Nisio semigroup convergence and

$$R(Q_n^x || P_n^x) \rightarrow \int_0^t L(x(s), u(s)) ds := I(x(\cdot))$$

and

$$V(t)f(x) = \sup_u \left\{ f(x(t)) - \int_0^t L(x(s), u(s)) ds : x(0) = x \right\}$$

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- In general, relaxed control (Young measure) replacing $u \dots$

One can completely formulate a program on co-tangent space (Hamiltonian equations) until the limit for representation ..

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- Then, by maximum principle,

$$Hf(x) = \sup_g (A^g f(x) - Lg(x))$$

$$Lg(x) = \sup_f (A^g f(x) - Hf(x)).$$

What's going on?

- Normalizing transition measure (g as a control)

$$P^g(t; x, dy) := \frac{e^{g(y)} P(t; x, dy)}{\int e^{g(z)} P(t; x, dz)}.$$

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$$\begin{aligned} A^g f(x) &:= \lim_{t \rightarrow 0+} E^{P_x^g} [f(X(t)) - f(X(0))] \\ &= \lim_{t \rightarrow 0+} \frac{E^x [e^{g(X(t))} (f(X(t)) - f(X(0)))]}{E^x [e^{g(X(t))}]} \\ &= e^{-g} A(fe^g)(x) - (e^{-g} f) A e^g(x); \\ Lg(x) &:= \lim_{t \rightarrow 0+} R(P^g(t; x, \cdot) \| P(t; x, \cdot)) = \dots \\ &= A^g g(x) - Hg(x). \end{aligned}$$

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- Embedding (reduction) of control space to R^d :

$$H_n f(x) = \sup_{u \in R^d} \left\{ \frac{1}{\sqrt{n}}\Delta f(x) + (b(x) + u) \cdot \nabla f(x) - \frac{1}{2}|u|^2 \right\}.$$

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- $Lg(x) = \lambda(x) \int ((g(y) - g(x))\frac{e^{g(y)}}{e^{g(x)}} - \frac{e^{g(y)}}{e^{g(x)}} + 1)\eta(x, dy)$
 $= \lambda(x)R(\eta^g(x, \cdot) || \eta(x, \cdot)).$

Nonlinear semigroup via viscosity solution language

Let

$$(I - \alpha H_n) f_n = h_n, \quad (I - \alpha H) f = h.$$

- Suppose we know that, if $h_n \rightarrow h$ and $H \subset \lim_n H_n$ (what sense?), then

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- Then

$$\begin{aligned} \lim_{n \rightarrow \infty} V_n(t) h_n &= \lim_n \lim_{m \rightarrow \infty} \left(I - \frac{t}{m} H_n \right)^{-m} h_n \\ &= \lim_m \lim_n \left(I - \frac{t}{m} H_n \right)^{-m} h_n = \lim_m \left(I - \frac{t}{m} H_n \right)^{-m} h \\ &= V(t) h \end{aligned}$$

What is viscosity solution in this context?

Problem: E is just a Polish space.

Solution: Use (nonlinear) maximum principle.

- $f, g \in M(E)$,

$$[f, g]_+ := \inf_{\epsilon > 0} \epsilon^{-1} (\|f + \epsilon g\| - \|f\|),$$

$$[f, g]_- := \sup_{\epsilon < 0} \epsilon^{-1} (\|f + \epsilon g\| - \|f\|).$$

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- $H \subset M(E) \times M(E)$ has maximum principle:

$$[f_1 - f_2, g_1 - g_2]_{\pm} \leq 0, \quad \forall (f_i, g_i) \in H;$$

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- If $\sup_x (f_0(x) - f(x)) \geq 0$, \Rightarrow super-solution...
- An exercise: Write down the precise expression ...

Barles-Perthame half relaxed limit, an abstract new proof

- Consider, for COMPACT E ,

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Q: Suppose $h_n \rightarrow h$ and $H_n \rightarrow H$, does $f_n \rightarrow f$?

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- Minty's device method by L.C.Evans:

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- Barles and Perthame:

$$\bar{f}(x) := \limsup_{n, y \rightarrow x} f_n(y), \quad \underline{f} := \liminf_{n, y \rightarrow x} f_n(y)$$

$$(I - \alpha H) \bar{f} \leq h, \quad (I - \alpha H) \underline{f} \geq h.$$

$f = \bar{f} = \underline{f}$? Answer: Comparison Principle.

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- Instead of

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$$\|f_n - f_0\| \leq \|f_n - \alpha H_n f_n - (f_0 - \alpha H_n f_0)\|,$$

- we want $\forall K \subset\subset E$ and $\epsilon > 0$, $\exists K_\epsilon \subset\subset E$, s.t.

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The noncompact E case

- Idea: Use concentration on compact set property of probability measures:

- Instead of

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- This a priori estimate comes from exponential tightness.
Why?

Exponential Tightness

Main ideas: stochastic generalization of Ascoli-Arzelà:

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- Exponential compact containment (construct Lyapunov funct.);
- Usually all follow from $\sup_n \sup_x H_n f_n < \infty$ for "wise" choices of f_n s.

First version (there are 2nd, 3rd, ...) of the LDP theorem

$\{X_n(\cdot) : n = 1, 2, \dots\}$ are E -valued processes that solve martingale problem with generator A_n .

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 $(I - \alpha H)\bar{f} \leq h$, and $(I - \alpha H)\underline{f} \geq h$ implies $\bar{f} \leq \underline{f}$.
- LDP holds for $\{X_n(\cdot) : n = 1, 2, \dots\}$.

The rate function

■ First,

$$I(x(\cdot)) = \sup_{t_0, t_1, \dots, t_k \in \Delta_x^c} \left\{ I_0(x(0)) + \sum_{i=1}^k I_{t_i - t_{i-1}}(x(t_i) | x(t_{i-1})) \right\}$$

where Δ_x^c is continuity time points of path $x(\cdot)$ and

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- Second, $\{V(t) : t \geq 0\}$ is a variational semigroup (the Nisio semigroup), so I should have a variational action integral representation.

The rate function and Nisio semigroup

Recall the infinitesimal entropy-pressure dual (always)

$$H_n f(x) = \sup_{g \in D(H)} \{A_n^g f(x) - L_n g(x)\}.$$

Idea: embed function space $D(H)$ into a nicer one.

- $A_n = \frac{1}{2n}\Delta$, then H_n is associated with $(U = D(H_n))$

$$dX_n = \nabla g(t, X_n(t))dt + \frac{1}{\sqrt{n}}dW_n,$$

$$L(X_n(t), g(t, X_n(t))) = \frac{1}{2}|\nabla g(t, X_n(t))|^2.$$

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- We could take $U = R^d$, $dX_n = udt + \frac{1}{\sqrt{n}}dW_n$ and $L(x, u) = \frac{1}{2}|u|^2$.

The rate function and Nisio semigroup - II

- Assume \exists metric space U , $Af : E \times U \mapsto R$ and $L : E \times U \mapsto R \cup \{+\infty\}$ s.t.

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- Γ_x means solving a controlled functional eqn

$$f(x(t)) - f(x(s)) = \int_s^t \int_U Af(x(r), u) \mu(du|r) dr, \quad \forall f \in D(\cdot)$$

The rate function and Nisio semigroup - III



$$I(x(\cdot)) = \inf \left\{ \int_0^T \int_U L(x(s), u) \mu(du|s) ds : \right. \\ \left. (x(\cdot), \mu(\cdot)) \in \Gamma_{x(0)} \right\}.$$

The rate function and Nisio semigroup - III



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- To rigorize above: comparison principle for $(I - \alpha H)f = h$.

What is the catch

- Classical LDP approach – analysis in path space (tangent space analysis)

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- Classical LDP approach – analysis in path space (tangent space analysis)
- This approach : More emphasize on analysis on function space (co-tangent space analysis).
- variations on path v.s. variations on test functions
- Technical subtlety: replace approximation for paths by proof of comparison principle for PDEs.